

$$\int \operatorname{ArcCoth}[a x]^n dx$$

■ **Reference:** CRC 586, A&S 4.6.48

■ **Derivation:** Integration by parts

■ **Rule:**

$$\int \operatorname{ArcCoth}[a x] dx \rightarrow x \operatorname{ArcCoth}[a x] + \frac{\operatorname{Log}[1 - a^2 x^2]}{2 a}$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_],x_Symbol] :=
  x*ArcCoth[a*x] + Log[1-a^2*x^2]/(2*a) /;
FreeQ[a,x]
```

■ **Derivation:** Integration by parts

■ **Rule:** If  $n \in \mathbb{Z} \wedge n > 1$ , then

$$\int \operatorname{ArcCoth}[a x]^n dx \rightarrow x \operatorname{ArcCoth}[a x]^n - a n \int \frac{x \operatorname{ArcCoth}[a x]^{n-1}}{1 - a^2 x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_,x_Symbol] :=
  x*ArcCoth[a*x]^n -
  Dist[a*n,Int[x*ArcCoth[a*x]^(n-1)/(1-a^2*x^2),x]] /;
FreeQ[a,x] && IntegerQ[n] && n>1
```

$$\int x^m \operatorname{ArcCoth}[a x]^n dx$$

- **Derivation:** Iterated integration by parts

- **Rule:** If  $n \in \mathbb{Z} \wedge n > 0$ , then

$$\int x \operatorname{ArcCoth}[a x]^n dx \rightarrow -\frac{\operatorname{ArcCoth}[a x]^n}{2 a^2} + \frac{x^2 \operatorname{ArcCoth}[a x]^n}{2} + \frac{n}{2 a} \int \operatorname{ArcCoth}[a x]^{n-1} dx$$

- **Program code:**

```
Int[x_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  -ArcCoth[a*x]^n/(2*a^2) + x^2*ArcCoth[a*x]^n/2 +
  Dist[n/(2*a),Int[ArcCoth[a*x]^(n-1),x]] /;
FreeQ[a,x] && IntegerQ[n] && n>0
```

- **Derivation:** Iterated integration by parts

- **Rule:** If  $n \in \mathbb{Z} \wedge n > 0 \wedge m > 1$ , then

$$\int x^m \operatorname{ArcCoth}[a x]^n dx \rightarrow -\frac{x^{m-1} \operatorname{ArcCoth}[a x]^n}{a^2 (m+1)} + \frac{x^{m+1} \operatorname{ArcCoth}[a x]^n}{m+1} +$$

$$\frac{n}{a (m+1)} \int x^{m-1} \operatorname{ArcCoth}[a x]^{n-1} dx + \frac{m-1}{a^2 (m+1)} \int x^{m-2} \operatorname{ArcCoth}[a x]^n dx$$

- **Program code:**

```
Int[x^m_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  -x^(m-1)*ArcCoth[a*x]^n/(a^2*(m+1)) + x^(m+1)*ArcCoth[a*x]^n/(m+1) +
  Dist[n/(a*(m+1)),Int[x^(m-1)*ArcCoth[a*x]^(n-1),x]] +
  Dist[(m-1)/(a^2*(m+1)),Int[x^(m-2)*ArcCoth[a*x]^n,x]] /;
FreeQ[a,x] && IntegerQ[n] && n>0 && RationalQ[m] && m>1
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $n \in \mathbb{Z} \wedge n > 1$ , then

$$\int \frac{\text{ArcCoth}[a x]^n}{x} dx \rightarrow 2 \text{ArcCoth}[a x]^n \text{ArcCoth}\left[1 - \frac{2}{1 - a x}\right] - 2 a n \int \frac{\text{ArcCoth}[a x]^{n-1} \text{ArcCoth}\left[1 - \frac{2}{1 - a x}\right]}{1 - a^2 x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_/x_,x_Symbol] :=
  2*ArcCoth[a*x]^n*ArcCoth[1-2/(1-a*x)] -
  Dist[2*a*n,Int[ArcCoth[a*x]^(n-1)*ArcCoth[1-2/(1-a*x)]/(1-a^2*x^2),x]] /;
FreeQ[a,x] && IntegerQ[n] && n>1
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{\text{ArcCoth}[a x]^n}{x^2} dx \rightarrow -\frac{\text{ArcCoth}[a x]^n}{x} + a n \int \frac{\text{ArcCoth}[a x]^{n-1}}{x (1 - a^2 x^2)} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_/x_^2,x_Symbol] :=
  -ArcCoth[a*x]^n/x +
  Dist[a*n,Int[ArcCoth[a*x]^(n-1)/(x*(1-a^2*x^2)),x]] /;
FreeQ[a,x] && IntegerQ[n] && n>0
```

■ **Derivation: Inverted iterated integration by parts**

■ **Rule:** If  $n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{\text{ArcCoth}[a x]^n}{x^3} dx \rightarrow \frac{a^2 \text{ArcCoth}[a x]^n}{2} - \frac{\text{ArcCoth}[a x]^n}{2 x^2} + \frac{a n}{2} \int \frac{\text{ArcCoth}[a x]^{n-1}}{x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_/x_^3,x_Symbol] :=
  a^2*ArcCoth[a*x]^n/2 - ArcCoth[a*x]^n/(2*x^2) +
  Dist[a*n/2,Int[ArcCoth[a*x]^(n-1)/x^2,x]] /;
FreeQ[a,x] && IntegerQ[n] && n>0
```

■ **Derivation: Inverted iterated integration by parts**

■ **Rule:** If  $n \in \mathbb{Z} \wedge n > 0 \wedge m < -3$ , then

$$\int x^m \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{x^{m+1} \operatorname{ArcCoth}[a x]^n}{m+1} - \frac{a^2 x^{m+3} \operatorname{ArcCoth}[a x]^n}{m+1} - \frac{a n}{m+1} \int x^{m+1} \operatorname{ArcCoth}[a x]^{n-1} dx + \frac{a^2 (m+3)}{m+1} \int x^{m+2} \operatorname{ArcCoth}[a x]^n dx$$

■ **Program code:**

```
Int[x_^m_*ArcCoth[a_*x_]^n_,x_Symbol] :=
  x^(m+1)*ArcCoth[a*x]^n/(m+1) - a^2*x^(m+3)*ArcCoth[a*x]^n/(m+1) -
  Dist[a*n/(m+1),Int[x^(m+1)*ArcCoth[a*x]^(n-1),x]] +
  Dist[a^2*(m+3)/(m+1),Int[x^(m+2)*ArcCoth[a*x]^n,x]] /;
FreeQ[a,x] && IntegerQ[n] && n>0 && RationalQ[m] && m<-3
```

$$\int \frac{\operatorname{ArcCoth}[a x]^n}{c + d x} dx$$

■ **Derivation:** Integration by parts

■ **Rule:** If  $a^2 c^2 = d^2 \wedge n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]^n}{c + d x} dx \rightarrow -\frac{\operatorname{ArcCoth}[a x]^n \operatorname{Log}\left[\frac{2c}{c+dx}\right]}{d} + \frac{a n}{d} \int \frac{\operatorname{ArcCoth}[a x]^{n-1} \operatorname{Log}\left[\frac{2c}{c+dx}\right]}{1 - a^2 x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_./(c_+d_.*x_),x_Symbol] :=
  -ArcCoth[a*x]^n*Log[2*c/(c+d*x)]/d +
  Dist[a*n/d,Int[ArcCoth[a*x]^(n-1)*Log[2*c/(c+d*x)]/(1-a^2*x^2),x] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c^2-d^2] && IntegerQ[n] && n>0
```

$$\int \frac{x^m \operatorname{ArcCoth}[a x]^n}{c + d x} dx$$

■ **Derivation:** Integration by parts

■ **Rule:** If  $a^2 c^2 = d^2 \wedge n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]^n}{x (c + d x)} dx \rightarrow \frac{\operatorname{ArcCoth}[a x]^n \operatorname{Log}\left[2 - \frac{2c}{c+dx}\right]}{c} - \frac{a n}{c} \int \frac{\operatorname{ArcCoth}[a x]^{n-1} \operatorname{Log}\left[2 - \frac{2c}{c+dx}\right]}{1 - a^2 x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_./(x_*(c_+d_.*x_)),x_Symbol] :=
  ArcCoth[a*x]^n*Log[2-2*c/(c+d*x)]/c -
  Dist[a*n/c,Int[ArcCoth[a*x]^(n-1)*Log[2-2*c/(c+d*x)]/(1-a^2*x^2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c^2-d^2] && IntegerQ[n] && n>0
```

■ **Derivation:** Integration by parts

■ **Rule:** If  $a^2 c^2 = d^2 \wedge n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]^n}{c x + d x^2} dx \rightarrow \frac{\operatorname{ArcCoth}[a x]^n \operatorname{Log}\left[2 - \frac{2c}{c+dx}\right]}{c} - \frac{a n}{c} \int \frac{\operatorname{ArcCoth}[a x]^{n-1} \operatorname{Log}\left[2 - \frac{2c}{c+dx}\right]}{1 - a^2 x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_./(c_.*x_+d_.*x_^2),x_Symbol] :=
  ArcCoth[a*x]^n*Log[2-2*c/(c+d*x)]/c -
  Dist[a*n/c,Int[ArcCoth[a*x]^(n-1)*Log[2-2*c/(c+d*x)]/(1-a^2*x^2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c^2-d^2] && IntegerQ[n] && n>0
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $\frac{x}{c+dx} = \frac{1}{d} - \frac{c}{d(c+dx)}$

■ **Rule:** If  $a^2 c^2 = d^2 \wedge m > 0 \wedge n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{x^m \operatorname{ArcCoth}[a x]^n}{c+dx} dx \rightarrow \frac{1}{d} \int x^{m-1} \operatorname{ArcCoth}[a x]^n dx - \frac{c}{d} \int \frac{x^{m-1} \operatorname{ArcCoth}[a x]^n}{c+dx} dx$$

■ **Program code:**

```
Int[x^m_*ArcCoth[a_*x_]^n_/ (c_+d_*x_), x_Symbol] :=
  Dist[1/d, Int[x^(m-1)*ArcCoth[a*x]^n, x]] -
  Dist[c/d, Int[x^(m-1)*ArcCoth[a*x]^n/(c+d*x), x]] /;
FreeQ[{a, c, d}, x] && ZeroQ[a^2*c^2-d^2] && RationalQ[m] && m>0 && IntegerQ[n] && n>0
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $\frac{1}{c+dx} = \frac{1}{c} - \frac{dx}{c(c+dx)}$

■ **Rule:** If  $a^2 c^2 = d^2 \wedge m < -1 \wedge n \in \mathbb{Z} \wedge n > 0$ , then

$$\int \frac{x^m \operatorname{ArcCoth}[a x]^n}{c+dx} dx \rightarrow \frac{1}{c} \int x^m \operatorname{ArcCoth}[a x]^n dx - \frac{d}{c} \int \frac{x^{m+1} \operatorname{ArcCoth}[a x]^n}{c+dx} dx$$

■ **Program code:**

```
Int[x^m_*ArcCoth[a_*x_]^n_/ (c_+d_*x_), x_Symbol] :=
  Dist[1/c, Int[x^m*ArcCoth[a*x]^n, x]] -
  Dist[d/c, Int[x^(m+1)*ArcCoth[a*x]^n/(c+d*x), x]] /;
FreeQ[{a, c, d}, x] && ZeroQ[a^2*c^2-d^2] && RationalQ[m] && m<-1 && IntegerQ[n] && n>0
```

$$\int \frac{\text{ArcCoth}[a x]^n}{c + d x^2} dx$$

- **Derivation: Reciprocal rule for integration**

- **Rule:** If  $a^2 c + d = 0$ , then

$$\int \frac{1}{(c + d x^2) \text{ArcCoth}[a x]} dx \rightarrow \frac{\text{Log}[\text{ArcCoth}[a x]]}{a c}$$

- **Program code:**

```
Int[1/((c+d_.*x_^2)*ArcCoth[a_.*x_]),x_Symbol] :=
  Log[ArcCoth[a*x]]/(a*c) /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d]
```

- **Derivation: Power rule for integration**

- **Rule:** If  $a^2 c + d = 0 \wedge n \neq -1$ , then

$$\int \frac{\text{ArcCoth}[a x]^n}{c + d x^2} dx \rightarrow \frac{\text{ArcCoth}[a x]^{n+1}}{a c (n+1)}$$

- **Program code:**

```
Int[ArcCoth[a_.*x_]^n_./(c+d_.*x_^2),x_Symbol] :=
  ArcCoth[a*x]^(n+1)/(a*c*(n+1)) /;
FreeQ[{a,c,d,n},x] && ZeroQ[a^2*c+d] && NonzeroQ[n+1]
```



$$\int \frac{x^m \operatorname{ArcCoth}[a x]^n}{c + d x^2} dx$$

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{x}{1-a^2 x^2} = -\frac{1}{a(1-a^2 x^2)} + \frac{1}{a(1-ax)}$

■ **Rule:** If  $a^2 c + d = 0 \wedge n > 0$ , then

$$\int \frac{x \operatorname{ArcCoth}[a x]^n}{c + d x^2} dx \rightarrow \frac{\operatorname{ArcCoth}[a x]^{n+1}}{d(n+1)} + \frac{1}{a c} \int \frac{\operatorname{ArcCoth}[a x]^n}{1-ax} dx$$

■ **Program code:**

```
Int[x_*ArcCoth[a_*x_]^n_/ (c_+d_*x_^2), x_Symbol] :=
  ArcCoth[a*x]^(n+1)/(d*(n+1)) +
  Dist[1/(a*c), Int[ArcCoth[a*x]^n/(1-a*x), x]] /;
FreeQ[{a,c,d}, x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0
```

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{1}{x(1-a^2 x^2)} = \frac{a}{1-a^2 x^2} + \frac{1}{x(1+ax)}$

■ **Rule:** If  $a^2 c + d = 0 \wedge n > 0$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]^n}{x(c + d x^2)} dx \rightarrow \frac{\operatorname{ArcCoth}[a x]^{n+1}}{c(n+1)} + \frac{1}{c} \int \frac{\operatorname{ArcCoth}[a x]^n}{x(1+ax)} dx$$

■ **Program code:**

```
Int[ArcCoth[a_*x_]^n_/ (x_*(c_+d_*x_^2)), x_Symbol] :=
  ArcCoth[a*x]^(n+1)/(c*(n+1)) +
  Dist[1/c, Int[ArcCoth[a*x]^n/(x*(1+a*x)), x]] /;
FreeQ[{a,c,d}, x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0
```

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{1}{x(1-a^2x^2)} = \frac{a}{1-a^2x^2} + \frac{1}{x(1+ax)}$

■ **Rule:** If  $a^2c + d = 0 \wedge n > 0$ , then

$$\int \frac{\text{ArcCoth}[ax]^n}{cx + dx^3} dx \rightarrow \frac{\text{ArcCoth}[ax]^{n+1}}{c(n+1)} + \frac{1}{c} \int \frac{\text{ArcCoth}[ax]^n}{x(1+ax)} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_./(c_.*x_+d_.*x_^3),x_Symbol] :=
  ArcCoth[a*x]^(n+1)/(c*(n+1)) +
  Dist[1/c,Int[ArcCoth[a*x]^n/(x*(1+a*x)),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0
```

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{x^2}{c+dx^2} = \frac{1}{d} - \frac{c}{d(c+dx^2)}$

■ **Rule:** If  $a^2c + d = 0 \wedge m > 1 \wedge n > 0$ , then

$$\int \frac{x^m \text{ArcCoth}[ax]^n}{c+dx^2} dx \rightarrow \frac{1}{d} \int x^{m-2} \text{ArcCoth}[ax]^n dx - \frac{c}{d} \int \frac{x^{m-2} \text{ArcCoth}[ax]^n}{c+dx^2} dx$$

■ **Program code:**

```
Int[x_^m_*ArcCoth[a_.*x_]^n_./(c_+d_.*x_^2),x_Symbol] :=
  Dist[1/d,Int[x^(m-2)*ArcCoth[a*x]^n,x]] -
  Dist[c/d,Int[x^(m-2)*ArcCoth[a*x]^n/(c+d*x^2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && m>1 && n>0
```

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{1}{c+dx^2} = \frac{1}{c} - \frac{dx^2}{c(c+dx^2)}$

■ **Rule:** If  $a^2c + d = 0 \wedge m < -1 \wedge n > 0$ , then

$$\int \frac{x^m \text{ArcCoth}[ax]^n}{c+dx^2} dx \rightarrow \frac{1}{c} \int x^m \text{ArcCoth}[ax]^n dx - \frac{d}{c} \int \frac{x^{m+2} \text{ArcCoth}[ax]^n}{c+dx^2} dx$$

■ **Program code:**

```
Int[x_^m_*ArcCoth[a_.*x_]^n_./(c_+d_.*x_^2),x_Symbol] :=
  Dist[1/c,Int[x^m*ArcCoth[a*x]^n,x]] -
  Dist[d/c,Int[x^(m+2)*ArcCoth[a*x]^n/(c+d*x^2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && m<-1 && n>0
```

■ **Derivation: Integration by substitution**

■ **Basis:** If  $m \in \mathbb{Z}$  or  $a > 0$ ,  $\frac{x^m \operatorname{ArcCoth}[a x]^n}{1-a^2 x^2} = \frac{\operatorname{Coth}[\operatorname{ArcCoth}[a x]]^m \operatorname{ArcCoth}[a x]^n}{a^{m+1}} \partial_x \operatorname{ArcCoth}[a x]$

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Q} \wedge (n < 0 \vee n \notin \mathbb{Z}) \wedge (m \in \mathbb{Z} \vee a > 0)$ , then

$$\int \frac{x^m \operatorname{ArcCoth}[a x]^n}{c + d x^2} dx \rightarrow \frac{1}{a^{m+1} c} \operatorname{Subst}\left[\int x^n \operatorname{Coth}[x]^m dx, x, \operatorname{ArcCoth}[a x]\right]$$

■ **Program code:**

```
Int[x_^m_*ArcCoth[a_*x_]^n_/(c_+d_*x_^2),x_Symbol] :=
  Dist[1/(a^(m+1)*c),Subst[Int[x^n*Coth[x]^m,x],x,ArcCoth[a*x]]]/;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && (n<0 || Not[IntegerQ[n]]) && (IntegerQ[m]
```

■ **Derivation: Integration by substitution**

■ **Basis:**  $\frac{x^m \operatorname{ArcCoth}[a x]^n}{1-a^2 x^2} = \frac{1}{a} \left( \frac{\operatorname{Coth}[\operatorname{ArcCoth}[a x]]}{a} \right)^m \operatorname{ArcCoth}[a x]^n \partial_x \operatorname{ArcCoth}[a x]$

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Q} \wedge (n < 0 \vee n \notin \mathbb{Z}) \wedge \neg (m \in \mathbb{Z} \vee a > 0)$ , then

$$\int \frac{x^m \operatorname{ArcCoth}[a x]^n}{c + d x^2} dx \rightarrow \frac{1}{a c} \operatorname{Subst}\left[\int x^n \left( \frac{\operatorname{Coth}[x]}{a} \right)^m dx, x, \operatorname{ArcCoth}[a x]\right]$$

■ **Program code:**

```
Int[x_^m_*ArcCoth[a_*x_]^n_/(c_+d_*x_^2),x_Symbol] :=
  Dist[1/(a*c),Subst[Int[x^n*(Coth[x]/a)^m,x],x,ArcCoth[a*x]]]/;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && (n<0 || Not[IntegerQ[n]]) && Not[IntegerQ[m]
```

$$\int \frac{\text{ArcCoth}[a x]^n \text{ArcCoth}[u]}{c + d x^2} dx$$

■ **Derivation:** Algebraic simplification

■ **Basis:**  $\text{ArcCoth}[z] = \frac{1}{2} \text{Log}\left[1 + \frac{1}{z}\right] - \frac{1}{2} \text{Log}\left[1 - \frac{1}{z}\right]$

■ **Rule:** If  $a^2 c + d = 0 \wedge n > 0 \wedge \left(u^2 = \left(1 - \frac{2}{1+ax}\right)^2 \vee u^2 = \left(1 - \frac{2}{1-ax}\right)^2\right)$ , then

$$\int \frac{\text{ArcCoth}[a x]^n \text{ArcCoth}[u]}{c + d x^2} dx \rightarrow \frac{1}{2} \int \frac{\text{ArcCoth}[a x]^n \text{Log}\left[1 + \frac{1}{u}\right]}{c + d x^2} dx - \frac{1}{2} \int \frac{\text{ArcCoth}[a x]^n \text{Log}\left[1 - \frac{1}{u}\right]}{c + d x^2} dx$$

■ **Program code:**

```
Int[ArcCoth[a_.*x_]^n_.*ArcCoth[u_] / (c_+d_.*x_^2), x_Symbol] :=
  Dist[1/2, Int[ArcCoth[a*x]^n*Log[Regularize[1+1/u,x]] / (c+d*x^2), x]] -
  Dist[1/2, Int[ArcCoth[a*x]^n*Log[Regularize[1-1/u,x]] / (c+d*x^2), x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0 && (ZeroQ[u^2-(1-2/(1+a*x))^2] || ZeroQ[u^2-
```

$$\int \frac{\text{ArcCoth}[a x]^n \text{Log}[u]}{c + d x^2} dx$$

■ Derivation: Integration by parts

- Rule: If  $a^2 c + d = 0 \wedge n > 0 \wedge (1-u)^2 = \left(1 - \frac{2}{1+ax}\right)^2$ , then

$$\int \frac{\text{ArcCoth}[a x]^n \text{Log}[u]}{c + d x^2} dx \rightarrow \frac{\text{ArcCoth}[a x]^n \text{PolyLog}[2, 1-u]}{2 a c} - \frac{n}{2} \int \frac{\text{ArcCoth}[a x]^{n-1} \text{PolyLog}[2, 1-u]}{c + d x^2} dx$$

■ Program code:

```
Int[ArcCoth[a_.*x_]^n_.*Log[u_] / (c_+d_.*x_^2), x_Symbol] :=
  ArcCoth[a*x]^n*PolyLog[2,1-u] / (2*a*c) -
  Dist[n/2, Int[ArcCoth[a*x]^(n-1)*PolyLog[2,1-u] / (c+d*x^2), x]] /;
FreeQ[{a,c,d}, x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0 && ZeroQ[(1-u)^2 - (1-2/(1+a*x))^2]
```

■ Derivation: Integration by parts

- Rule: If  $a^2 c + d = 0 \wedge n > 0 \wedge (1-u)^2 = \left(1 - \frac{2}{1-ax}\right)^2$ , then

$$\int \frac{\text{ArcCoth}[a x]^n \text{Log}[u]}{c + d x^2} dx \rightarrow -\frac{\text{ArcCoth}[a x]^n \text{PolyLog}[2, 1-u]}{2 a c} + \frac{n}{2} \int \frac{\text{ArcCoth}[a x]^{n-1} \text{PolyLog}[2, 1-u]}{c + d x^2} dx$$

■ Program code:

```
Int[ArcCoth[a_.*x_]^n_.*Log[u_] / (c_+d_.*x_^2), x_Symbol] :=
  -ArcCoth[a*x]^n*PolyLog[2,1-u] / (2*a*c) +
  Dist[n/2, Int[ArcCoth[a*x]^(n-1)*PolyLog[2,1-u] / (c+d*x^2), x]] /;
FreeQ[{a,c,d}, x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0 && ZeroQ[(1-u)^2 - (1-2/(1-a*x))^2]
```

$$\int \frac{\text{ArcCoth}[a x]^n \text{PolyLog}[p, u]}{c + d x^2} dx$$

■ Derivation: Integration by parts

■ Rule: If  $a^2 c + d = 0 \wedge n > 0 \wedge u^2 = \left(1 - \frac{2}{1+ax}\right)^2$ , then

$$\int \frac{\text{ArcCoth}[a x]^n \text{PolyLog}[p, u]}{c + d x^2} dx \rightarrow -\frac{\text{ArcCoth}[a x]^n \text{PolyLog}[p+1, u]}{2 a c} + \frac{n}{2} \int \frac{\text{ArcCoth}[a x]^{n-1} \text{PolyLog}[p+1, u]}{c + d x^2} dx$$

■ Program code:

```
Int[ArcCoth[a_.x_]^n_.PolyLog[p_,u_]/(c_+d_.x_^2),x_Symbol] :=
  -ArcCoth[a*x]^n*PolyLog[p+1,u]/(2*a*c) +
  Dist[n/2,Int[ArcCoth[a*x]^(n-1)*PolyLog[p+1,u]/(c+d*x^2),x]] /;
FreeQ[{a,c,d,p},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0 && ZeroQ[u^2-(1-2/(1+a*x))^2]
```

■ Derivation: Integration by parts

■ Rule: If  $a^2 c + d = 0 \wedge n > 0 \wedge u^2 = \left(1 - \frac{2}{1-ax}\right)^2$ , then

$$\int \frac{\text{ArcCoth}[a x]^n \text{PolyLog}[p, u]}{c + d x^2} dx \rightarrow \frac{\text{ArcCoth}[a x]^n \text{PolyLog}[p+1, u]}{2 a c} - \frac{n}{2} \int \frac{\text{ArcCoth}[a x]^{n-1} \text{PolyLog}[p+1, u]}{c + d x^2} dx$$

■ Program code:

```
Int[ArcCoth[a_.x_]^n_.PolyLog[p_,u_]/(c_+d_.x_^2),x_Symbol] :=
  ArcCoth[a*x]^n*PolyLog[p+1,u]/(2*a*c) -
  Dist[n/2,Int[ArcCoth[a*x]^(n-1)*PolyLog[p+1,u]/(c+d*x^2),x]] /;
FreeQ[{a,c,d,p},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>0 && ZeroQ[u^2-(1-2/(1-a*x))^2]
```

$$\int \frac{\operatorname{ArcTanh}[a x]^m \operatorname{ArcCoth}[a x]^n}{c + d x^2} dx$$

- **Derivation:** Integration by parts

- **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Z} \wedge 0 < n < m$ , then

$$\int \frac{\operatorname{ArcTanh}[a x]^m \operatorname{ArcCoth}[a x]^n}{c + d x^2} dx \rightarrow \frac{\operatorname{ArcTanh}[a x]^{m+1} \operatorname{ArcCoth}[a x]^n}{a c (m+1)} - \frac{n}{m+1} \int \frac{\operatorname{ArcTanh}[a x]^{m+1} \operatorname{ArcCoth}[a x]^{n-1}}{c + d x^2} dx$$

- **Program code:**

```
Int[ArcTanh[a_*x_]^m_*ArcCoth[a_*x_]^n_/(c_+d_*x_^2),x_Symbol] :=
  ArcTanh[a*x]^(m+1)*ArcCoth[a*x]^n/(a*c*(m+1)) -
  Dist[n/(m+1),Int[ArcTanh[a*x]^(m+1)*ArcCoth[a*x]^(n-1)/(c+d*x^2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n] && 0<n<m
```

$$\int (c + d x^2)^m \operatorname{ArcCoth}[a x]^n dx$$

- Rule: If  $a^2 c + d = 0 \wedge c > 0$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]}{\sqrt{c + d x^2}} dx \rightarrow -\frac{2 \operatorname{ArcCoth}[a x] \operatorname{ArcTan}\left[\frac{\sqrt{1-a x}}{\sqrt{1+a x}}\right]}{a \sqrt{c}} - \frac{i \operatorname{PolyLog}\left[2, -\frac{i \sqrt{1-a x}}{\sqrt{1+a x}}\right]}{a \sqrt{c}} + \frac{i \operatorname{PolyLog}\left[2, \frac{i \sqrt{1-a x}}{\sqrt{1+a x}}\right]}{a \sqrt{c}}$$

- Program code:

```
Int[ArcCoth[a_.x_]/Sqrt[c_+d_.x_^2],x_Symbol] :=
-2*ArcCoth[a*x]*ArcTan[Sqrt[1-a*x]/Sqrt[1+a*x]]/(a*Sqrt[c]) -
I*PolyLog[2,-I*Sqrt[1-a*x]/Sqrt[1+a*x]]/(a*Sqrt[c]) +
I*PolyLog[2,I*Sqrt[1-a*x]/Sqrt[1+a*x]]/(a*Sqrt[c]) /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && PositiveQ[c]
```

- Basis:  $\partial_x \frac{\sqrt{1-a^2 x^2}}{\sqrt{c - c a^2 x^2}} = 0$

- Rule: If  $a^2 c + d = 0 \wedge \neg (c > 0)$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]}{\sqrt{c + d x^2}} dx \rightarrow \frac{\sqrt{1 - a^2 x^2}}{\sqrt{c + d x^2}} \int \frac{\operatorname{ArcCoth}[a x]}{\sqrt{1 - a^2 x^2}} dx$$

- Program code:

```
Int[ArcCoth[a_.x_]/Sqrt[c_+d_.x_^2],x_Symbol] :=
Sqrt[1-a^2*x^2]/Sqrt[c+d*x^2]*Int[ArcCoth[a*x]/Sqrt[1-a^2*x^2],x] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && Not[PositiveQ[c]]
```

- Rule: If  $a^2 c + d = 0$ , then

$$\int \frac{\operatorname{ArcCoth}[a x]}{(c + d x^2)^{3/2}} dx \rightarrow -\frac{1}{a c \sqrt{c + d x^2}} + \frac{x \operatorname{ArcCoth}[a x]}{c \sqrt{c + d x^2}}$$

- Program code:

```
Int[ArcCoth[a_.x_]/(c_+d_.x_^2)^(3/2),x_Symbol] :=
-1/(a*c*Sqrt[c+d*x^2]) +
x*ArcCoth[a*x]/(c*Sqrt[c+d*x^2]) /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d]
```



- Rule: If  $a^2 c + d = 0 \wedge n > 1$ , then

$$\int \frac{\text{ArcCoth}[a x]^n}{(c + d x^2)^{3/2}} dx \rightarrow -\frac{n \text{ArcCoth}[a x]^{n-1}}{a c \sqrt{c + d x^2}} + \frac{x \text{ArcCoth}[a x]^n}{c \sqrt{c + d x^2}} + n(n-1) \int \frac{\text{ArcCoth}[a x]^{n-2}}{(c + d x^2)^{3/2}} dx$$

- Program code:

```
Int[ArcCoth[a_.x_]^n_/(c_+d_.x_^2)^(3/2),x_Symbol] :=
  -n*ArcCoth[a*x]^(n-1)/(a*c*Sqrt[c+d*x^2]) +
  x*ArcCoth[a*x]^n/(c*Sqrt[c+d*x^2]) +
  Dist[n*(n-1),Int[ArcCoth[a*x]^(n-2)/(c+d*x^2)^(3/2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n>1
```

- Rule: If  $a^2 c + d = 0 \wedge n < -1 \wedge n \neq -2$ , then

$$\int \frac{\text{ArcCoth}[a x]^n}{(c + d x^2)^{3/2}} dx \rightarrow \frac{\text{ArcCoth}[a x]^{n+1}}{a c (n+1) \sqrt{c + d x^2}} - \frac{x \text{ArcCoth}[a x]^{n+2}}{c (n+1) (n+2) \sqrt{c + d x^2}} + \frac{1}{(n+1) (n+2)} \int \frac{\text{ArcCoth}[a x]^{n+2}}{(c + d x^2)^{3/2}} dx$$

- Program code:

```
Int[ArcCoth[a_.x_]^n_/(c_+d_.x_^2)^(3/2),x_Symbol] :=
  ArcCoth[a*x]^(n+1)/(a*c*(n+1)*Sqrt[c+d*x^2]) -
  x*ArcCoth[a*x]^(n+2)/(c*(n+1)*(n+2)*Sqrt[c+d*x^2]) +
  Dist[1/((n+1)*(n+2)),Int[ArcCoth[a*x]^(n+2)/(c+d*x^2)^(3/2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n<-1 && n≠-2
```

- Rule: If  $a^2 c + d = 0 \wedge m > 0$ , then

$$\int (c + d x^2)^m \text{ArcCoth}[a x] dx \rightarrow \frac{(c + d x^2)^m}{2 a m (2 m + 1)} + \frac{x (c + d x^2)^m \text{ArcCoth}[a x]}{(2 m + 1)} + \frac{2 c m}{2 m + 1} \int (c + d x^2)^{m-1} \text{ArcCoth}[a x] dx$$

- Program code:

```
Int[(c_+d_.x_^2)^m_.*ArcCoth[a_.x_],x_Symbol] :=
  (c+d*x^2)^m/(2*a*m*(2*m+1)) +
  x*(c+d*x^2)^m*ArcCoth[a*x]/(2*m+1) +
  Dist[2*c*m/(2*m+1),Int[(c+d*x^2)^(m-1)*ArcCoth[a*x],x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[m] && m>0
```

- Rule: If  $a^2 c + d = 0 \bigwedge m < -1 \bigwedge m \neq -\frac{3}{2}$ , then

$$\int (c + d x^2)^m \operatorname{ArcCoth}[a x] dx \rightarrow -\frac{(c + d x^2)^{m+1}}{4 a c (m+1)^2} - \frac{x (c + d x^2)^{m+1} \operatorname{ArcCoth}[a x]}{2 c (m+1)} + \frac{2 m + 3}{2 c (m+1)} \int (c + d x^2)^{m+1} \operatorname{ArcCoth}[a x] dx$$

- Program code:

```
Int[(c+d_.x^2)^m_*ArcCoth[a_.x_],x_Symbol] :=
  -(c+d*x^2)^(m+1)/(4*a*c*(m+1)^2) -
  x*(c+d*x^2)^(m+1)*ArcCoth[a*x]/(2*c*(m+1)) +
  Dist[(2*m+3)/(2*c*(m+1)),Int[(c+d*x^2)^(m+1)*ArcCoth[a*x],x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[m] && m<-1 && m!=3/2
```

- Rule: If  $a^2 c + d = 0 \bigwedge m < -1 \bigwedge m \neq -\frac{3}{2} \bigwedge n > 1$ , then

$$\int (c + d x^2)^m \operatorname{ArcCoth}[a x]^n dx \rightarrow -\frac{n (c + d x^2)^{m+1} \operatorname{ArcCoth}[a x]^{n-1}}{4 a c (m+1)^2} - \frac{x (c + d x^2)^{m+1} \operatorname{ArcCoth}[a x]^n}{2 c (m+1)} + \frac{2 m + 3}{2 c (m+1)} \int (c + d x^2)^{m+1} \operatorname{ArcCoth}[a x]^n dx + \frac{n (n-1)}{4 (m+1)^2} \int (c + d x^2)^m \operatorname{ArcCoth}[a x]^{n-2} dx$$

- Program code:

```
Int[(c+d_.x^2)^m_*ArcCoth[a_.x_]^n_,x_Symbol] :=
  -n*(c+d*x^2)^(m+1)*ArcCoth[a*x]^(n-1)/(4*a*c*(m+1)^2) -
  x*(c+d*x^2)^(m+1)*ArcCoth[a*x]^n/(2*c*(m+1)) +
  Dist[(2*m+3)/(2*c*(m+1)),Int[(c+d*x^2)^(m+1)*ArcCoth[a*x]^n,x]] +
  Dist[n*(n-1)/(4*(m+1)^2),Int[(c+d*x^2)^m*ArcCoth[a*x]^(n-2),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && m<-1 && m!=3/2 && n>1
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $a^2 c + d = 0 \wedge m < -1 \wedge n < -1$ , then

$$\int (c + d x^2)^m \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{(c + d x^2)^{m+1} \operatorname{ArcCoth}[a x]^{n+1}}{a c (n+1)} + \frac{2 a (m+1)}{n+1} \int x (c + d x^2)^m \operatorname{ArcCoth}[a x]^{n+1} dx$$

■ **Program code:**

```
Int[(c_+d_.*x_^2)^m_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  (c+d*x^2)^(m+1)*ArcCoth[a*x]^(n+1)/(a*c*(n+1)) +
  Dist[2*a*(m+1)/(n+1),Int[x*(c+d*x^2)^m_*ArcCoth[a*x]^(n+1),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && m<-1 && n<-1
```

■ **Derivation: Integration by substitution**

■ **Basis:** If  $m \in \mathbb{Z}$ ,  $(1 - a^2 x^2)^m \operatorname{ArcCoth}[a x]^n = -\frac{(-1)^m}{a} \operatorname{Csch}[\operatorname{ArcCoth}[a x]]^{2(m+1)} \operatorname{ArcCoth}[a x]^n \partial_x \operatorname{ArcCoth}[a x]$

■ **Rule:** If  $a^2 c + d = 0 \wedge m \in \mathbb{Z} \wedge n \in \mathbb{Q} \wedge m < -1 \wedge (n < 0 \vee n \notin \mathbb{Z})$ , then

$$\int (c + d x^2)^m \operatorname{ArcCoth}[a x]^n dx \rightarrow -\frac{(-c)^m}{a} \operatorname{Subst}\left[\int x^n \operatorname{Csch}[x]^{2(m+1)} dx, x, \operatorname{ArcCoth}[a x]\right]$$

■ **Program code:**

```
Int[(c_+d_.*x_^2)^m_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  -Dist[(-c)^m/a,Subst[Int[x^n*Csch[x]^(2*(m+1)),x],x,ArcCoth[a*x]]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegerQ[m] && RationalQ[n] && m<-1 && (n<0 || Not[IntegerQ[n]])
```

■ **Basis:** If  $a^2 c + d = 0$ ,  $D\left[\frac{c^{\frac{m-1}{2}} \sqrt{c+d x^2}}{\sqrt{1-a^2 x^2}}, x\right] = 0$

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Q} \wedge m < -1 \wedge (n < 0 \vee n \notin \mathbb{Z}) \wedge m - \frac{1}{2} \in \mathbb{Z} \wedge \neg (c > 0)$ , then

$$\int (c + d x^2)^m \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{c^{m-\frac{1}{2}} \sqrt{c+d x^2}}{\sqrt{1-a^2 x^2}} \int (1 - a^2 x^2)^m \operatorname{ArcCoth}[a x]^n dx$$

■ **Program code:**

```
(* Int[(c_+d_.*x_^2)^m_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  c^(m-1/2)*Sqrt[c+d*x^2]/Sqrt[1-a^2*x^2]*Int[(1-a^2*x^2)^m_*ArcCoth[a*x]^n,x] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && m<-1 && (n<0 || Not[IntegerQ[n]]) && IntegerQ[m-1/2]
```

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx$$

■ **Derivation:** Integration by parts

■ **Rule:** If  $a^2 c + d = 0 \wedge p \in \mathbb{Q} \wedge n > 0 \wedge p \neq -1$ , then

$$\int x (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{(c + d x^2)^{p+1} \operatorname{ArcCoth}[a x]^n}{2 d (p+1)} + \frac{n}{2 a (p+1)} \int (c + d x^2)^p \operatorname{ArcCoth}[a x]^{n-1} dx$$

■ **Program code:**

```
Int[x*(c+d.*x^2)^p_.*ArcCoth[a.*x_]^n_,x_Symbol] :=
  (c+d*x^2)^(p+1)*ArcCoth[a*x]^n/(2*d*(p+1)) +
  Dist[n/(2*a*(p+1)),Int[(c+d*x^2)^p*ArcCoth[a*x]^(n-1),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{n,p}] && n>0 && p≠-1
```

■ **Rule:** If  $a^2 c + d = 0 \wedge p \in \mathbb{Q}$ , then

$$\int \frac{x (c + d x^2)^p}{\operatorname{ArcCoth}[a x]^2} dx \rightarrow -\frac{x (c + d x^2)^{p+1}}{a c \operatorname{ArcCoth}[a x]} + \frac{1}{a} \int \frac{(1 - (2 p + 3) a^2 x^2) (c + d x^2)^p}{\operatorname{ArcCoth}[a x]} dx$$

■ **Program code:**

```
Int[x*(c+d.*x^2)^p_. / ArcCoth[a.*x_]^2,x_Symbol] :=
  -x*(c+d*x^2)^(p+1)/(a*c*ArcCoth[a*x]) +
  Dist[1/a,Int[(1-(2*p+3)*a^2*x^2)*(c+d*x^2)^p/ArcCoth[a*x],x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[p]
```

■ **Rule:** If  $a^2 c + d = 0 \wedge n < -1 \wedge n \neq -2$ , then

$$\int \frac{x \operatorname{ArcCoth}[a x]^n}{(c + d x^2)^2} dx \rightarrow \frac{x \operatorname{ArcCoth}[a x]^{n+1}}{a c (n+1) (c + d x^2)} + \frac{(1 + a^2 x^2) \operatorname{ArcCoth}[a x]^{n+2}}{d (n+1) (n+2) (c + d x^2)} + \frac{4}{(n+1) (n+2)} \int \frac{x \operatorname{ArcCoth}[a x]^{n+2}}{(c + d x^2)^2} dx$$

■ **Program code:**

```
Int[x*ArcCoth[a.*x_]^n_/(c+d.*x^2)^2,x_Symbol] :=
  x*ArcCoth[a*x]^(n+1)/(a*c*(n+1)*(c+d*x^2)) +
  (1+a^2*x^2)*ArcCoth[a*x]^(n+2)/(d*(n+1)*(n+2)*(c+d*x^2)) +
  Dist[4/((n+1)*(n+2)),Int[x*ArcCoth[a*x]^(n+2)/(c+d*x^2)^2,x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[n] && n<-1 && n≠-2
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n, 2p \in \mathbb{Z} \wedge m < -1 \wedge n > 0 \wedge m + 2p + 3 = 0$ , then

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{x^{m+1} (c + d x^2)^{p+1} \operatorname{ArcCoth}[a x]^n}{c (m+1)} - \frac{a n}{m+1} \int x^{m+1} (c + d x^2)^p \operatorname{ArcCoth}[a x]^{n-1} dx$$

■ **Program code:**

```
Int[x_^m*(c_+d_*x_^2)^p_*ArcCoth[a_*x_]^n_,x_Symbol] :=
  x^(m+1)*(c+d*x^2)^(p+1)*ArcCoth[a*x]^n/(c*(m+1)) -
  Dist[a*n/(m+1),Int[x^(m+1)*(c+d*x^2)^p*ArcCoth[a*x]^(n-1),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n,2*p] && m<-1 && n>0 && ZeroQ[m+2*p+3]
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n, 2p \in \mathbb{Z} \wedge n < -1 \wedge m + 2p + 2 = 0$ , then

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{x^m (c + d x^2)^{p+1} \operatorname{ArcCoth}[a x]^{n+1}}{a c (n+1)} - \frac{m}{a (n+1)} \int x^{m-1} (c + d x^2)^p \operatorname{ArcCoth}[a x]^{n+1} dx$$

■ **Program code:**

```
Int[x_^m*(c_+d_*x_^2)^p_*ArcCoth[a_*x_]^n_,x_Symbol] :=
  x^m*(c+d*x^2)^(p+1)*ArcCoth[a*x]^(n+1)/(a*c*(n+1)) -
  Dist[m/(a*(n+1)),Int[x^(m-1)*(c+d*x^2)^p*ArcCoth[a*x]^(n+1),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n,2*p] && n<-1 && ZeroQ[m+2*p+2]
```

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{x^2}{c+dx^2} = \frac{1}{d} - \frac{c}{d(c+dx^2)}$

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n, 2p \in \mathbb{Z} \wedge m > 1 \wedge n \neq -1 \wedge p < -1$ , then

$$\int x^m (c + dx^2)^p \operatorname{ArcCoth}[ax]^n dx \rightarrow \frac{1}{d} \int x^{m-2} (c + dx^2)^{p+1} \operatorname{ArcCoth}[ax]^n dx - \frac{c}{d} \int x^{m-2} (c + dx^2)^p \operatorname{ArcCoth}[ax]^n dx$$

■ **Program code:**

```
Int[x_^m*(c_+d_.*x_^2)^p*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  Dist[1/d,Int[x^(m-2)*(c+d*x^2)^(p+1)*ArcCoth[a*x]^n,x]] -
  Dist[c/d,Int[x^(m-2)*(c+d*x^2)^p*ArcCoth[a*x]^n,x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n,2*p] && m>1 && n≠-1 && p<-1
```

■ **Derivation: Algebraic expansion**

■ **Basis:**  $\frac{1}{c+dx^2} = \frac{1}{c} - \frac{dx^2}{c(c+dx^2)}$

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n, 2p \in \mathbb{Z} \wedge m < 0 \wedge n \neq -1 \wedge p < -1$ , then

$$\int x^m (c + dx^2)^p \operatorname{ArcCoth}[ax]^n dx \rightarrow \frac{1}{c} \int x^m (c + dx^2)^{p+1} \operatorname{ArcCoth}[ax]^n dx - \frac{d}{c} \int x^{m+2} (c + dx^2)^p \operatorname{ArcCoth}[ax]^n dx$$

■ **Program code:**

```
Int[x_^m*(c_+d_.*x_^2)^p*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  Dist[1/c,Int[x^m*(c+d*x^2)^(p+1)*ArcCoth[a*x]^n,x]] -
  Dist[d/c,Int[x^(m+2)*(c+d*x^2)^p*ArcCoth[a*x]^n,x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n,2*p] && m<0 && n≠-1 && p<-1
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n, 2p \in \mathbb{Z} \wedge m < -1 \wedge n > 0 \wedge m + 2p + 3 \neq 0$ , then

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{x^{m+1} (c + d x^2)^{p+1} \operatorname{ArcCoth}[a x]^n}{c (m+1)} - \frac{a n}{m+1} \int x^{m+1} (c + d x^2)^p \operatorname{ArcCoth}[a x]^{n-1} dx + \frac{a^2 (m+2p+3)}{m+1} \int x^{m+2} (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx$$

■ **Program code:**

```
Int[x_^m.*(c_+d_.*x_^2)^p_.*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  x^(m+1)*(c+d*x^2)^(p+1)*ArcCoth[a*x]^n/(c*(m+1)) -
  Dist[a*n/(m+1),Int[x^(m+1)*(c+d*x^2)^p*ArcCoth[a*x]^(n-1),x]] +
  Dist[a^2*(m+2*p+3)/(m+1),Int[x^(m+2)*(c+d*x^2)^p*ArcCoth[a*x]^n,x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n,2*p] && m<-1 && n>0 && NonzeroQ[m+2*p+3]
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n, 2p \in \mathbb{Z} \wedge n < -1 \wedge m + 2p + 2 \neq 0$ , then

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{x^m (c + d x^2)^{p+1} \operatorname{ArcCoth}[a x]^{n+1}}{a c (n+1)} - \frac{m}{a (n+1)} \int x^{m-1} (c + d x^2)^p \operatorname{ArcCoth}[a x]^{n+1} dx + \frac{a (m+2p+2)}{n+1} \int x^{m+1} (c + d x^2)^p \operatorname{ArcCoth}[a x]^{n+1} dx$$

■ **Program code:**

```
Int[x_^m.*(c_+d_.*x_^2)^p_.*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  x^m*(c+d*x^2)^(p+1)*ArcCoth[a*x]^(n+1)/(a*c*(n+1)) -
  Dist[m/(a*(n+1)),Int[x^(m-1)*(c+d*x^2)^p*ArcCoth[a*x]^(n+1),x]] +
  Dist[a*(m+2*p+2)/(n+1),Int[x^(m+1)*(c+d*x^2)^p*ArcCoth[a*x]^(n+1),x]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && IntegersQ[m,n,2*p] && n<-1 && NonzeroQ[m+2*p+2]
```

■ **Derivation: Integration by substitution**

- **Basis:** If  $p \in \mathbb{Z}$  and  $(m \in \mathbb{Z} \text{ or } a > 0)$ ,  
 $(e + f x^m) (1 - a^2 x^2)^p \operatorname{ArcCoth}[a x]^n =$   

$$- \frac{(-1)^p}{a^{m+1}} (e a^m + f \operatorname{Coth}[\operatorname{ArcCoth}[a x]]^m) \operatorname{Csch}[\operatorname{ArcCoth}[a x]]^{2(p+1)} \operatorname{ArcCoth}[a x]^n \partial_x \operatorname{ArcCoth}[a x]$$

- **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Q} \wedge p \in \mathbb{Z} \wedge p < -1 \wedge (n < 0 \vee n \notin \mathbb{Z}) \wedge (m \in \mathbb{Z} \vee a > 0)$ , then

$$\int (e + f x^m) (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow - \frac{(-c)^p}{a^{m+1}}$$

$$\operatorname{Subst} \left[ \int x^n (e a^m + f \operatorname{Coth}[x]^m) \operatorname{Csch}[x]^{2(p+1)} dx, x, \operatorname{ArcCoth}[a x] \right]$$

■ **Program code:**

```
Int[(e_+f_.*x_^m_)*(c_+d_.*x_^2)^p_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  -Dist[(-c)^p/a^(m+1),Subst[Int[Expand[x^n*TrigReduce[Regularize[(e*a^m+f*Coth[x]^m)*Csch[x]^(2*(p+1))],x],x,ArcCoth[a*x]]],x);
FreeQ[{a,c,d,e,f},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && IntegerQ[p] && p<-1 && (n<0 || Not[IntegerQ[n]])
```

■ **Derivation: Integration by substitution**

- **Basis:** If  $p \in \mathbb{Z}$ ,  
 $x^m (1 - a^2 x^2)^p \operatorname{ArcCoth}[a x]^n = - \frac{(-1)^p}{a} \left( \frac{\operatorname{Coth}[\operatorname{ArcCoth}[a x]]}{a} \right)^m \operatorname{Csch}[\operatorname{ArcCoth}[a x]]^{2(p+1)} \operatorname{ArcCoth}[a x]^n \partial_x \operatorname{ArcCoth}[a x]$

- **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Q} \wedge p \in \mathbb{Z} \wedge p < -1 \wedge (n < 0 \vee n \notin \mathbb{Z}) \wedge \neg (m \in \mathbb{Z} \vee a > 0)$ , then

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow - \frac{(-c)^p}{a} \operatorname{Subst} \left[ \int x^n (\operatorname{Coth}[x] / a)^m \operatorname{Csch}[x]^{2(p+1)} dx, x, \operatorname{ArcCoth}[a x] \right]$$

■ **Program code:**

```
Int[x_^m_.*(c_+d_.*x_^2)^p_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  -Dist[(-c)^p/a,Subst[Int[x^n*(Coth[x]/a)^m*Csch[x]^(2*(p+1)),x],x,ArcCoth[a*x]]] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n}] && IntegerQ[p] && p<-1 && (n<0 || Not[IntegerQ[n]])
```



■ **Basis:** If  $a^2 c + d = 0$ ,  $D\left[\frac{c^{\frac{p-1}{2}} \sqrt{c+dx^2}}{\sqrt{1-a^2 x^2}}, x\right] = 0$

■ **Rule:** If  $a^2 c + d = 0 \wedge m, n \in \mathbb{Q} \wedge p < -1 \wedge (n < 0 \vee n \notin \mathbb{Z}) \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge \neg (c > 0)$ , then

$$\int x^m (c + d x^2)^p \operatorname{ArcCoth}[a x]^n dx \rightarrow \frac{c^{p-\frac{1}{2}} \sqrt{c + d x^2}}{\sqrt{1 - a^2 x^2}} \int x^m (1 - a^2 x^2)^p \operatorname{ArcCoth}[a x]^n dx$$

■ **Program code:**

```
(* Int[x_^m_.*(c+d_.**x_^2)^p_*ArcCoth[a_.*x_]^n_,x_Symbol] :=
  c^(p-1/2)*Sqrt[c+d*x^2]/Sqrt[1-a^2*x^2]*Int[x^m*(1-a^2*x^2)^p*ArcCoth[a*x]^n,x] /;
FreeQ[{a,c,d},x] && ZeroQ[a^2*c+d] && RationalQ[{m,n,p}] && p<-1 && (n<0 || Not[IntegerQ[n]]) && Int
```

$$\int \operatorname{ArcCoth}[a + b x^n] \, dx$$

■ **Reference:** CRC 586, A&S 4.6.48

■ **Derivation:** Integration by parts

■ **Rule:**

$$\int \operatorname{ArcCoth}[a + b x] \, dx \rightarrow \frac{(a + b x) \operatorname{ArcCoth}[a + b x]}{b} + \frac{\operatorname{Log}[1 - (a + b x)^2]}{2 b}$$

■ **Program code:**

```
Int[ArcCoth[a_+b_.*x_],x_Symbol] :=
  (a+b*x)*ArcCoth[a+b*x]/b + Log[1-(a+b*x)^2]/(2*b) /;
FreeQ[{a,b},x]
```

■ **Reference:** CRC 586, A&S 4.6.48

■ **Derivation:** Integration by parts

■ **Rule:** If  $n \in \mathbb{Q}$ , then

$$\int \operatorname{ArcCoth}[a + b x^n] \, dx \rightarrow x \operatorname{ArcCoth}[a + b x^n] - b n \int \frac{x^n}{1 - a^2 - 2 a b x^n - b^2 x^{2n}} \, dx$$

■ **Program code:**

```
Int[ArcCoth[a_+b_.*x_^n_],x_Symbol] :=
  x*ArcCoth[a+b*x^n] -
  Dist[b*n,Int[x^n/(1-a^2-2*a*b*x^n-b^2*x^(2*n)),x]] /;
FreeQ[{a,b},x] && RationalQ[n]
```

$$\int x^m \operatorname{ArcCoth}[a + b x^n] \, dx$$

- **Derivation:** Algebraic expansion

- **Basis:**  $\operatorname{ArcCoth}[z] = \frac{1}{2} \operatorname{Log}\left[1 + \frac{1}{z}\right] - \frac{1}{2} \operatorname{Log}\left[1 - \frac{1}{z}\right]$

- **Rule:**

$$\int \frac{\operatorname{ArcCoth}[a + b x^n]}{x} \, dx \rightarrow \frac{1}{2} \int \frac{\operatorname{Log}\left[1 + \frac{1}{a + b x^n}\right]}{x} \, dx - \frac{1}{2} \int \frac{\operatorname{Log}\left[1 - \frac{1}{a + b x^n}\right]}{x} \, dx$$

- **Program code:**

```
Int[ArcCoth[a_+b_.*x_^n_]/x_,x_Symbol] :=
  Dist[1/2,Int[Log[1+1/(a+b*x^n)]/x,x]] -
  Dist[1/2,Int[Log[1-1/(a+b*x^n)]/x,x]] /;
FreeQ[{a,b,n},x]
```

- **Reference:** CRC 590, A&S 4.6.60

- **Derivation:** Integration by parts

- **Rule:** If  $m, n \in \mathbb{Q} \wedge m+1 \neq 0 \wedge m+1 \neq n$ , then

$$\int x^m \operatorname{ArcCoth}[a + b x^n] \, dx \rightarrow \frac{x^{m+1} \operatorname{ArcCoth}[a + b x^n]}{m+1} - \frac{b n}{m+1} \int \frac{x^{m+n}}{1 - a^2 - 2 a b x^n - b^2 x^{2n}} \, dx$$

- **Program code:**

```
Int[x_^m_.*ArcCoth[a_+b_.*x_^n_],x_Symbol] :=
  x^(m+1)*ArcCoth[a+b*x^n]/(m+1) -
  Dist[b*n/(m+1),Int[x^(m+n)/(1-a^2-2*a*b*x^n-b^2*x^(2*n)),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m+1!=0 && m+1!=n
```

$$\int \operatorname{ArcCoth}[a + b x]^n dx$$

- **Derivation:** Integration by substitution

- **Rule:** If  $n \in \mathbb{Z} \wedge n > 1$ , then

$$\int \operatorname{ArcCoth}[a + b x]^n dx \rightarrow \frac{1}{b} \operatorname{Subst}\left[\int \operatorname{ArcCoth}[x]^n dx, x, a + b x\right]$$

- **Program code:**

```
Int[ArcCoth[a_+b_.*x_]^n_,x_Symbol] :=
  Dist[1/b,Subst[Int[ArcCoth[x]^n,x],x,a+b*x]] /;
FreeQ[{a,b},x] && IntegerQ[n] && n>1
```

$$\int x^m \operatorname{ArcCoth}[a + b x]^n dx$$

- **Derivation:** Integration by substitution

- **Rule:** If  $m, n \in \mathbb{Z} \wedge m > 0 \wedge n > 1$ , then

$$\int x^m \operatorname{ArcCoth}[a + b x]^n dx \rightarrow \frac{1}{b^{m+1}} \operatorname{Subst}\left[\int (x - a)^m \operatorname{ArcCoth}[x]^n dx, x, a + b x\right]$$

- **Program code:**

```
Int[x_^m_.*ArcCoth[a_+b_.*x_]^n_,x_Symbol] :=
  Dist[1/b^(m+1),Subst[Int[(x-a)^m*ArcCoth[x]^n,x],x,a+b*x]] /;
FreeQ[{a,b},x] && IntegersQ[m,n] && m>0 && n>1
```

$$\int \frac{\text{ArcCoth}[a + b x]}{c + d x^n} dx$$

■ **Derivation:** Algebraic simplification

■ **Basis:**  $\text{ArcCoth}[z] = \frac{1}{2} \text{Log}\left[1 + \frac{1}{z}\right] - \frac{1}{2} \text{Log}\left[1 - \frac{1}{z}\right]$

■ **Rule:** If  $n \in \mathbb{Z} \wedge \neg (n = 2 \wedge b^2 c + d = 0)$ , then

$$\int \frac{\text{ArcCoth}[b x]}{c + d x^n} dx \rightarrow \frac{1}{2} \int \frac{\text{Log}\left[1 + \frac{1}{b x}\right]}{c + d x^n} dx - \frac{1}{2} \int \frac{\text{Log}\left[1 - \frac{1}{b x}\right]}{c + d x^n} dx$$

■ **Program code:**

```
Int[ArcCoth[b_.*x_]/(c_+d_.*x_^n_),x_Symbol] :=
  Dist[1/2,Int[Log[1+1/(b*x)]/(c+d*x^n),x]] -
  Dist[1/2,Int[Log[1-1/(b*x)]/(c+d*x^n),x]] /;
FreeQ[{b,c,d},x] && IntegerQ[n] && Not[n==2 && ZeroQ[b^2*c+d]]
```

■ **Derivation:** Algebraic simplification

■ **Basis:**  $\text{ArcCoth}[z] = \frac{1}{2} \text{Log}\left[1 + \frac{1}{z}\right] - \frac{1}{2} \text{Log}\left[1 - \frac{1}{z}\right]$

■ **Rule:** If  $n \in \mathbb{Z} \wedge \neg (n = 1 \wedge a d - b c = 0)$ , then

$$\int \frac{\text{ArcCoth}[a + b x]}{c + d x^n} dx \rightarrow \frac{1}{2} \int \frac{\text{Log}\left[1 + \frac{1}{a + b x}\right]}{c + d x^n} dx - \frac{1}{2} \int \frac{\text{Log}\left[1 - \frac{1}{a + b x}\right]}{c + d x^n} dx$$

■ **Program code:**

```
Int[ArcCoth[a_+b_.*x_]/(c_+d_.*x_^n_),x_Symbol] :=
  Dist[1/2,Int[Log[1+1/(a+b*x)]/(c+d*x^n),x]] -
  Dist[1/2,Int[Log[1-1/(a+b*x)]/(c+d*x^n),x]] /;
FreeQ[{a,b,c,d},x] && IntegerQ[n] && Not[n==1 && ZeroQ[a*d-b*c]]
```

$$\int u \operatorname{ArcCoth} \left[ \frac{c}{a + b x^n} \right]^m dx$$

■ **Derivation:** Algebraic simplification

■ **Basis:**  $\operatorname{ArcCoth}[z] = \operatorname{ArcTanh}\left[\frac{1}{z}\right]$

■ **Rule:**

$$\int u \operatorname{ArcCoth} \left[ \frac{c}{a + b x^n} \right]^m dx \rightarrow \int u \operatorname{ArcTanh} \left[ \frac{a}{c} + \frac{b x^n}{c} \right]^m dx$$

■ **Program code:**

```
Int[u_.*ArcCoth[c_./(a_.+b_.*x^n_.)]^m_.,x_Symbol] :=
  Int[u*ArcTanh[a/c+b*x^n/c]^m,x] /;
FreeQ[{a,b,c,n,m},x]
```

$$\int \frac{f[x, \operatorname{ArcCoth}[a + b x]]}{1 - (a + b x)^2} dx$$

■ **Derivation: Integration by substitution**

■ **Basis:**  $\frac{f[z]}{1-z^2} = f[\operatorname{Coth}[\operatorname{ArcCoth}[z]]] \operatorname{ArcCoth}'[z]$

■ **Basis:**  $r + s x + t x^2 = -\frac{s^2 - 4 r t}{4 t} \left(1 - \frac{(s + 2 t x)^2}{s^2 - 4 r t}\right)$

■ **Basis:**  $1 - \operatorname{Coth}[z]^2 = -\operatorname{Csch}[z]^2$

■ **Rule:**

$$\int \frac{f[x, \operatorname{ArcCoth}[a + b x]]}{1 - (a + b x)^2} dx \rightarrow \frac{1}{b} \operatorname{Subst}\left[\int f\left[-\frac{a}{b} + \frac{\operatorname{Coth}[x]}{b}, x\right] dx, x, \operatorname{ArcCoth}[a + b x]\right]$$

■ **Program code:**

```
If[ShowSteps,

Int[u_*v_^n_,x_Symbol] :=
Module[{tmp=InverseFunctionOfLinear[u,x]},
ShowStep["", "Int[f[x,ArcCoth[a+b*x]]/(1-(a+b*x)^2),x]",
"Subst[Int[f[-a/b+Coth[x]/b,x],x,ArcCoth[a+b*x]]/b",Hold[
Dist[(-Discriminant[v,x]/(4*Coefficient[v,x,2]))^n/Coefficient[tmp[[1]],x,1],
Subst[Int[Regularize[SubstForInverseFunction[u,tmp,x]*(-Csch[x]^2)^(n+1),x],x], x, tmp]]] /;
NotFalseQ[tmp] && Head[tmp]==ArcCoth && ZeroQ[Discriminant[v,x]*tmp[[1]]^2-D[v,x]^2] /;
SimplifyFlag && QuadraticQ[v,x] && IntegerQ[n] && n<0 && PosQ[Discriminant[v,x]] && MatchQ[u,r_.*f_^

Int[u_*v_^n_,x_Symbol] :=
Module[{tmp=InverseFunctionOfLinear[u,x]},
Dist[(-Discriminant[v,x]/(4*Coefficient[v,x,2]))^n/Coefficient[tmp[[1]],x,1],
Subst[Int[Regularize[SubstForInverseFunction[u,tmp,x]*(-Csch[x]^2)^(n+1),x],x], x, tmp]] /;
NotFalseQ[tmp] && Head[tmp]==ArcCoth && ZeroQ[Discriminant[v,x]*tmp[[1]]^2-D[v,x]^2] /;
QuadraticQ[v,x] && IntegerQ[n] && n<0 && PosQ[Discriminant[v,x]] && MatchQ[u,r_.*f_^w_ /; FreeQ[f,x]
```



$$\int u e^{n \operatorname{ArcCoth}[v]} dx$$

■ **Derivation: Algebraic simplification**

■ **Basis:** If  $\frac{n}{2} \in \mathbb{Z}$ ,  $e^{n \operatorname{ArcCoth}[z]} = \frac{(1+z)^{n/2}}{(-1+z)^{n/2}}$

■ **Rule:** If  $\frac{n}{2} \in \mathbb{Z}$ , then

$$\int u e^{n \operatorname{ArcCoth}[v]} dx \rightarrow \int \frac{u (1+v)^{n/2}}{(-1+v)^{n/2}} dx$$

■ **Program code:**

```
Int[u_.*E^(n_.*ArcCoth[v_]),x_Symbol] :=
  Int[u*(1+v)^(n/2)/(-1+v)^(n/2),x] /;
EvenQ[n]
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $e^{n \operatorname{ArcCoth}[z]} = \left( \frac{1}{\sqrt{1-\frac{1}{z^2}}} + \frac{1}{z \sqrt{1-\frac{1}{z^2}}} \right)^n$

■ **Rule:** If  $\frac{n-1}{2} \in \mathbb{Z}$ , then

$$\int e^{n \operatorname{ArcCoth}[v]} dx \rightarrow \int \left( \frac{1}{\sqrt{1-\frac{1}{v^2}}} + \frac{1}{x \sqrt{1-\frac{1}{v^2}}} \right)^n dx$$

■ **Program code:**

```
Int[E^(n_.*ArcCoth[v_]),x_Symbol] :=
  Int[Expand[(1/Sqrt[1-1/v^2] + 1/(x*Sqrt[1-1/v^2]))^n],x] /;
OddQ[n]
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $e^{n \operatorname{ArcCoth}[z]} = \left( \frac{1}{\sqrt{1 - \frac{1}{z^2}}} + \frac{1}{z \sqrt{1 - \frac{1}{z^2}}} \right)^n$

■ **Rule:** If  $m \in \mathbb{Q} \wedge \frac{n-1}{2} \in \mathbb{Z}$ , then

$$\int x^m e^{n \operatorname{ArcCoth}[v]} dx \rightarrow \int x^m \left( \frac{1}{\sqrt{1 - \frac{1}{v^2}}} + \frac{1}{x \sqrt{1 - \frac{1}{v^2}}} \right)^n dx$$

■ **Program code:**

```
Int[x_^m_.*E^(n_.*ArcCoth[v_]),x_Symbol] :=
  Int[Expand[x^m*(1/Sqrt[1-1/v^2] + 1/(x*Sqrt[1-1/v^2]))^n],x] /;
RationalQ[m] && OddQ[n]
```

■ **Derivation: Integration by parts**

■ **Rule:**

$$\int e^{\operatorname{ArcCoth}[a+bx]/2} dx \rightarrow x e^{\operatorname{ArcCoth}[a+bx]/2} - \frac{b}{2} \int \frac{x e^{\operatorname{ArcCoth}[a+bx]/2}}{1 - (a+bx)^2} dx$$

■ **Program code:**

```
Int[E^(ArcCoth[a_.+b_.*x_]/2), x_Symbol] :=
  x*E^(ArcCoth[a+b*x]/2) -
  Dist[b/2,Int[x*E^(ArcCoth[a+b*x]/2)/(1-(a+b*x)^2),x] /;
FreeQ[{a,b},x]
```

■ **Derivation: Integration by parts**

■ **Rule:** If  $m \in \mathbb{Z} \wedge m > 0$ , then

$$\int x^m e^{\operatorname{ArcCoth}[a+bx]/2} dx \rightarrow \frac{x^{m+1} e^{\operatorname{ArcCoth}[a+bx]/2}}{m+1} - \frac{b}{2(m+1)} \int \frac{x^{m+1} e^{\operatorname{ArcCoth}[a+bx]/2}}{1 - (a+bx)^2} dx$$

■ **Program code:**

```
Int[x_^m_.*E^(ArcCoth[a_.+b_.*x_]/2), x_Symbol] :=
  x^(m+1)*E^(ArcCoth[a+b*x]/2)/(m+1) -
  Dist[b/(2*(m+1)),Int[x^(m+1)*E^(ArcCoth[a+b*x]/2)/(1-(a+b*x)^2),x] /;
FreeQ[{a,b},x] && IntegerQ[m] && m>0
```

■ **Derivation: Algebraic simplification**

■ **Basis:** If  $\frac{n-1}{2} \in \mathbb{Z}$ ,  $e^{n \operatorname{ArcCoth}[z]} (1 - z^2)^m = - \frac{(-1)^{\frac{n-1}{2}} z \sqrt{1 - \frac{1}{z^2}} (1-z)^{m-\frac{n}{2}} (1+z)^{m+\frac{n}{2}}}{\sqrt{1-z^2}}$

■ **Basis:**  $\partial_x \frac{f[x] \sqrt{a - \frac{a}{f[x]^2}}}{\sqrt{1-f[x]^2}} = 0$

■ **Rule:** If  $\frac{n-1}{2} \in \mathbb{Z} \wedge m - \frac{1}{2} \in \mathbb{Z}$ , then

$$\int u e^{n \operatorname{ArcCoth}[v]} (1 - v^2)^m dx \rightarrow - \frac{(-1)^{\frac{n-1}{2}} v \sqrt{1 - \frac{1}{v^2}}}{\sqrt{1 - v^2}} \int u (1 - v)^{m-\frac{n}{2}} (1 + v)^{m+\frac{n}{2}} dv$$

■ **Program code:**

```
Int[u_.*E^(n_.*ArcCoth[v_])*(1-v_^2)^m_.,x_Symbol] :=
  -(-1)^( (n-1)/2)*v*Sqrt[1-1/v^2]/Sqrt[1-v^2]*Int[u*(1-v)^(m-n/2)*(1+v)^(m+n/2),x] /;
OddQ[n] && IntegerQ[m-1/2]
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $e^{\operatorname{ArcCoth}[z]} (1 - z^2) = -z (1 + z) \sqrt{1 - \frac{1}{z^2}}$

■ **Rule:** If  $m \in \mathbb{Z} \wedge m > 0$ , then

$$\int u e^{\operatorname{ArcCoth}[v]} (a - a v^2)^m dx \rightarrow -a^m \int u v (1 + v) \sqrt{1 - \frac{1}{v^2}} (1 - v^2)^{m-1} dv$$

■ **Program code:**

```
Int[u_.*E^ArcCoth[v_]*(a_+b_.*v_^2)^m_.,x_Symbol] :=
  -Dist[a^m,Int[Expand[u*v*(1+v)*Sqrt[1-1/v^2]*(1-v^2)^(m-1),x],x]] /;
FreeQ[{a,b},x] && ZeroQ[a+b] && IntegerQ[m] && m>0
```

■ **Basis:**  $\partial_x \frac{(a - f[x]^2)^m}{(1 - f[x]^2)^m} = 0$

■ **Rule:** If  $a \neq 1$ , then

$$\int u e^{\text{ArcCoth}[v]} (a - a v^2)^m dx \rightarrow \frac{(a - a v^2)^m}{(1 - v^2)^m} \int u e^{\text{ArcCoth}[v]} (1 - v^2)^m dx$$

■ **Program code:**

```
Int[u_.*E^ArcCoth[v_]*(a_+b_.*v_^2)^m_.,x_Symbol] :=
  (a+b*v^2)^m/(1-v^2)^m*Int[u*E^ArcCoth[v]*(1-v^2)^m,x] /;
FreeQ[{a,b},x] && ZeroQ[a+b] && NonzeroQ[a-1]
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $e^{n \text{ArcCoth}[z]} = \frac{\left(1 + \frac{1}{z}\right)^n}{\left(1 - \frac{1}{z^2}\right)^{n/2}}$

■ **Rule:** If  $n \in \mathbb{Z} \wedge m - \frac{n}{2} \in \mathbb{Z}$ , then

$$\int u e^{n \text{ArcCoth}[v]} \left(1 - \frac{1}{v^2}\right)^m dx \rightarrow \int \frac{u (1+v)^{m+\frac{n}{2}} (-1+v)^{m-\frac{n}{2}}}{v^{2m}} dx$$

■ **Program code:**

```
Int[u_.*E^(n_.*ArcCoth[v_])*(1-1/v_^2)^m_.,x_Symbol] :=
  Int[u*(1+v)^(m+n/2)*(-1+v)^(m-n/2)/v^(2*m),x] /;
IntegerQ[n] && IntegerQ[m-n/2]
```

■ **Derivation: Algebraic simplification**

■ **Basis:**  $e^{n \text{ArcCoth}[z]} = \frac{\left(1 + \frac{1}{z}\right)^n}{\left(1 - \frac{1}{z^2}\right)^{n/2}}$

■ **Rule:** If  $n \in \mathbb{Q} \wedge 2m \in \mathbb{Z}$ , then

$$\int u e^{n \text{ArcCoth}[v]} \left(1 - \frac{1}{v^2}\right)^m dx \rightarrow \int u \left(1 + \frac{1}{v}\right)^n \left(1 - \frac{1}{v^2}\right)^{m-\frac{n}{2}} dx$$

■ **Program code:**

```
Int[u_.*E^(n_.*ArcCoth[v_])*(1-1/v_^2)^m_.,x_Symbol] :=
  Int[Expand[u*(1+1/v)^n*(1-1/v^2)^(m-n/2),x],x] /;
RationalQ[n] && IntegerQ[2*m]
```

■ **Derivation: Piecewise constant extraction and algebraic simplification**

■ **Basis:**  $e^{\text{ArcCoth}[z]} = \frac{1 + \frac{1}{z}}{\sqrt{1 - \frac{1}{z^2}}}$

■ **Basis:**  $\partial_x \frac{\left(a - \frac{a}{f(x)^2}\right)^m}{\left(1 - \frac{1}{f(x)^2}\right)^m} = 0$

■ **Rule:** If  $2m \in \mathbb{Z}$ , then

$$\int u e^{\text{ArcCoth}[v]} \left(a - \frac{a}{v^2}\right)^m dx \rightarrow \frac{\left(a - \frac{a}{v^2}\right)^m}{\left(1 - \frac{1}{v^2}\right)^m} \int u \left(1 + \frac{1}{v}\right) \left(1 - \frac{1}{v^2}\right)^{m-\frac{1}{2}} dx$$

■ **Program code:**

```
Int[u_.*E^ArcCoth[v_]*(a_+b_/v_^2)^m_.,x_Symbol] :=
  (a+b/v^2)^m/(1-1/v^2)^m*Int[u*(1+1/v)*(1-1/v^2)^(m-1/2),x] /;
FreeQ[{a,b},x] && ZeroQ[a+b] && IntegerQ[2*m]
```

$$\int \operatorname{ArcCoth}\left[b f^{c+d x}\right] d x$$

- **Derivation:** Algebraic simplification

- **Basis:**  $\operatorname{ArcCoth}[z] = \frac{1}{2} \operatorname{Log}\left[1 + \frac{1}{z}\right] - \frac{1}{2} \operatorname{Log}\left[1 - \frac{1}{z}\right]$

- **Rule:**

$$\int \operatorname{ArcCoth}\left[b f^{c+d x}\right] d x \rightarrow \frac{1}{2} \int \operatorname{Log}\left[1 + \frac{1}{b f^{c+d x}}\right] d x - \frac{1}{2} \int \operatorname{Log}\left[1 - \frac{1}{b f^{c+d x}}\right] d x$$

- **Program code:**

```
Int[ArcCoth[b_.*f_^(c_.+d_.*x_)],x_Symbol] :=
  Dist[1/2,Int[Log[1+1/(b*f^(c+d*x))],x]] -
  Dist[1/2,Int[Log[1-1/(b*f^(c+d*x))],x]] /;
FreeQ[{b,c,d,f},x]
```

$$\int x^m \operatorname{ArcCoth}\left[b f^{c+dx}\right] dx$$

- **Derivation:** Algebraic simplification

- **Basis:**  $\operatorname{ArcCoth}[z] = \frac{1}{2} \operatorname{Log}\left[1 + \frac{1}{z}\right] - \frac{1}{2} \operatorname{Log}\left[1 - \frac{1}{z}\right]$

- **Rule:** If  $m \in \mathbb{Z} \wedge m > 0$ , then

$$\int x^m \operatorname{ArcCoth}\left[b f^{c+dx}\right] dx \rightarrow \frac{1}{2} \int x^m \operatorname{Log}\left[1 + \frac{1}{b f^{c+dx}}\right] dx - \frac{1}{2} \int x^m \operatorname{Log}\left[1 - \frac{1}{b f^{c+dx}}\right] dx$$

- **Program code:**

```
Int[x_^m_*ArcCoth[b_*f^(c_+d_*x_)],x_Symbol] :=
  Dist[1/2,Int[x^m*Log[1+1/(b*f^(c+d*x))],x]] -
  Dist[1/2,Int[x^m*Log[1-1/(b*f^(c+d*x))],x]] /;
FreeQ[{b,c,d,f},x] && IntegerQ[m] && m>0
```

$$\int v \operatorname{ArcCoth}[u] \, dx$$

- **Derivation:** Integration by parts

- **Rule:** If  $u$  is free of inverse functions, then

$$\int \operatorname{ArcCoth}[u] \, dx \rightarrow x \operatorname{ArcCoth}[u] - \int \frac{x \partial_x u}{1 - u^2} \, dx$$

- **Program code:**

```
Int[ArcCoth[u_], x_Symbol] :=
  x*ArcCoth[u] -
  Int[Regularize[x*D[u,x]/(1-u^2), x], x] /;
InverseFunctionFreeQ[u, x]
```

- **Derivation:** Integration by parts

- **Rule:** If  $m + 1 \neq 0 \wedge u$  is free of inverse functions, then

$$\int x^m \operatorname{ArcCoth}[u] \, dx \rightarrow \frac{x^{m+1} \operatorname{ArcCoth}[u]}{m+1} - \frac{1}{m+1} \int \frac{x^{m+1} \partial_x u}{1 - u^2} \, dx$$

- **Program code:**

```
Int[x_^m_.*ArcCoth[u_], x_Symbol] :=
  x^(m+1)*ArcCoth[u]/(m+1) -
  Dist[1/(m+1), Int[Regularize[x^(m+1)*D[u,x]/(1-u^2), x], x]] /;
FreeQ[m, x] && NonzeroQ[m+1] && InverseFunctionFreeQ[u, x] &&
  Not[FunctionOfQ[x^(m+1), u, x]] &&
  FalseQ[PowerVariableExpn[u, m+1, x]]
```



- **Derivation: Integration by parts**

- **Rule:** If  $u$  is free of inverse functions, let  $w = \int v \, dx$ , if  $w$  is free of inverse functions, then

$$\int v \operatorname{ArcCoth}[u] \, dx \rightarrow w \operatorname{ArcCoth}[u] - \int \frac{w \partial_x u}{1 - u^2} \, dx$$

- **Program code:**

```
Int[v_*ArcCoth[u_],x_Symbol] :=
  Module[{w=Block[{ShowSteps=False,StepCounter=None}, Int[v,x]]},
    w*ArcCoth[u] -
    Int[Regularize[w*D[u,x]/(1-u^2),x],x] /;
    InverseFunctionFreeQ[w,x] /;
    InverseFunctionFreeQ[u,x] &&
    Not[MatchQ[v, x^m_. /; FreeQ[m,x]]] &&
    FalseQ[FunctionOfLinear[v*ArcCoth[u],x]]
```